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Stratified Type Theory

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Type Universes

```
Inductive T : Type :=
| mkT : forall X : Type, (X -> T) -> T.
```

Type Universes

```
Inductive T : Type :=
| mkT : forall X : Type, (X -> T) -> T.
```

```
Definition TT : T := mkT T (fun x => x).
```

```
The term "T" has type "Type@{T.u0+1}" while it is expected to have type "Type@{T.u0}"
(universe inconsistency: Cannot enforce T.u0 < T.u0 because T.u0 = T.u0).
```

Universe Hierarchy

$$\frac{\texttt{j < k}}{\vdash \texttt{Type@\{j\} : Type@\{k\}}}$$

Universe Hierarchy

$$\frac{j < k}{\vdash \star@\{j\} : \star@\{k\}}$$

Universe Hierarchy

$$\frac{j < k}{\vdash \star@\{j\} : \star@\{k\}}$$

$$\frac{}{\nvdash \star@\{k\} : \star@\{k\}}$$

inconsistent!

Logical Inconsistency

Inductive False : Type.

Definition (consistency): $\nexists b$ s.t. $\vdash b : \mathrm{False}$

Girard's paradox: $\vdash \star : \star$ is inconsistent

Universe Level Polymorphism

```
Polymorphic Definition id@{u} (X : Type@{u}) (x : X) := x.

Polymorphic Inductive T@{u v} : Type@{v} :=
| mkT : forall X : Type@{u}, (X -> T) -> T.
(* u v |= u < v *)
```

🤔 do we have to stratify universes into a hierarchy?

💡 stratify **judgements** instead of **universes**

Stratified Type Theory

functions | universes | cumulativity | ...

$\boxed{\Gamma \vdash a : A}$ *conventional TT*

$$\frac{\Gamma \vdash A : \star@\{k\} \qquad \Gamma, x : A \vdash B : \star@\{k\}}{\Gamma \vdash \Pi x : A.\ B : \star@\{k\}}$$

$\boxed{\Gamma \vdash a : A \ @k}$ *StraTT*

$$\frac{\Gamma \vdash A : \star \ @j \qquad \Gamma, x : A \ @j \vdash B : \star \ @k \qquad j < k}{\Gamma \vdash \Pi x : A \ @j.\ B : \star \ @k}$$

Stratified Type Theory

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$$\frac{}{\Gamma \vdash \star : \star\ @k}$$

Stratified Type Theory

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$\boxed{\Gamma \vdash a : A}$ *conventional TT*

$$\frac{\Gamma \vdash A : \star@\{j\} \qquad j \leq k}{\Gamma \vdash A : \star@\{k\}}$$

$$\frac{j < k}{\Gamma \vdash \star@\{j\} : \star@\{k\}}$$

$\boxed{\Gamma \vdash a : A \; @k}$ *StraTT*

$$\frac{\Gamma \vdash a : A \; @j \qquad j \leq k}{\Gamma \vdash a : A \; @k}$$

$$\frac{}{\Gamma \vdash \star : \star \; @k}$$

Stratified Type Theory

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$\boxed{\Gamma \vdash a : A\ @k}$ *StraTT*

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$$\frac{}{\Gamma \vdash \star : \star\ @k}$$

Confidence Check: identity function

Rocq

```
Universes v.
Definition id (X : Type@{v}) (x : X) : X := x.
```

StraTT

```
id : ΠX : ⋆ @1. Πx : X @1. X @2
id X x ≔ x
                @1
```

Confidence Check: identity function applied to itself

Rocq

```
Universes u v. Constraint u < v.
Definition id (X : Type@{v}) (x : X) : X := x.
Definition idid : forall X : Type@{u}, X -> X :=
id (forall X : Type@{u}, X -> X) (fun X x => id X x).
                                      Type@{u}    Type@{v}
```

StraTT

```
id : ΠX : ⋆ @1. Πx : X @1. X @2
id X x ≔ x
```

Confidence Check: identity function applied to itself

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Definition id (X : Type@{v}) (x : X) : X := x.
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id (forall X : Type@{u}, X -> X) (fun X x => id X x).
```

StraTT

$$\mathrm{id} : \Pi X : \star\ @1.\ \underbrace{\Pi x : X\ @1}_{@1}.\ X\ @2$$

$$\mathrm{id}\ X\ x \coloneqq x$$

$$\mathrm{idid} : \Pi X : \star\ @0.\ \Pi x : X\ @0.\ X\ @2$$

$$\mathrm{idid} \coloneqq \mathrm{id}\ (\Pi X : \star\ @0.\ \Pi x : X\ @0.\ X)\ \overbrace{\cancel{(\lambda X.\ \lambda x.\ \mathrm{id}\ X\ x)}}^{@2}$$

Confidence Check: identity function applied to itself

Rocq

```
Universes u v. Constraint u < v.
Definition id (X : Type@{v}) (x : X) : X := x.
Definition idid : forall X : Type@{u}, X -> X :=
id (forall X : Type@{u}, X -> X) (fun X x => id X x).
```

StraTT

```
id : ΠX : ⋆ @1. X → X @2
id X x ≔ x        @2

idid : ΠX : ⋆ @0. X → X @2                    @2
idid ≔ id (ΠX : ⋆ @0. X → X) (λX. λx. id X x)
```

Stratified Type Theory

functions | universes | cumulativity | …

$\boxed{\Gamma \vdash a : A}$ *conventional TT*

$$\frac{\Gamma \vdash A : \star@\{k\} \qquad \Gamma, x : A \vdash B : \star@\{k\}}{\Gamma \vdash \Pi x : A.\ B : \star@\{k\}}$$

$$\frac{\Gamma \vdash A : \star@\{k\} \qquad \Gamma \vdash B : \star@\{k\}}{\Gamma \vdash A \to B : \star@\{k\}}$$

$\boxed{\Gamma \vdash a : A\ @k}$ *StraTT*

dependent functions

$$\frac{\Gamma \vdash A : \star\ @j \qquad \Gamma, x : A\ @j \vdash B : \star\ @k \qquad j < k}{\Gamma \vdash \Pi x : A\ @j.\ B : \star\ @k}$$

"floating" functions

$$\frac{\Gamma \vdash A : \star\ @k \qquad \Gamma \vdash A : \star\ @k}{\Gamma \vdash A \to B : \star\ @k}$$

Stratified Type Theory

functions | universes | cumulativity | ...

$$\boxed{\Gamma \vdash a : A \ @k} \quad \textit{StraTT}$$

$$\frac{\Gamma \vdash a : A \ @j \qquad j \leq k}{\Gamma \vdash a : A \ @k}$$

"floating" functions

$$\frac{\Gamma \vdash A : \star \ @k \qquad \Gamma \vdash A : \star \ @k}{\Gamma \vdash A \to B : \star \ @k}$$

$$\frac{j \leq k \qquad \Gamma \vdash f : \star \to \star \ @j}{\Gamma \vdash f : \star \to \star \ @k}$$

↳ takes ★ @k, not ★ @j

is StraTT useable?

implementation with extensions:

- prototype type checker
  - displacement
  - datatypes
  - annotation inference
- small core library

Displacement

original

```
id : ΠX : ⋆ @0. Πx : X @0. X @1
id X x ≔ x
```

duplicate

```
id' : ΠX : ⋆ @1. Πx : X @1. X @2
id' X x ≔ x
```

```
idid : ΠX : ⋆ @0. X → X @2
idid ≔ id' (ΠX : ⋆ @0. X → X) (λX. λx. id' X x)
```

Displacement

original `id : ΠX : ⋆ @0. Πx : X @0. X @1`
`id X x ≔ x`

~~`id' : ΠX : ⋆ @1. Πx : X @1. X @2`~~
~~`id' X x ≔ x`~~

displaced $\texttt{id}^{+1}$ `: ΠX : ⋆ @1. Πx : X @1. X @2`

`idid : ΠX : ⋆ @0. X → X @2`
`idid ≔` $\texttt{id}^{+1}$ `(ΠX : ⋆ @0. X → X) (λX. λx.` $\texttt{id}^{+1}$ `X x)`

Level Annotation Inference

```
unannot. idid : ΠX : ⋆ @█. X → X @█
  StraTT idid ≔ id■ (ΠX : ⋆ @█. X → X) (λX. λx. id■ X x)

inferred idid : ΠX : ⋆ @0. X → X @2
  StraTT idid ≔ id⁺¹ (ΠX : ⋆ @0. X → X) (λX. λx. id⁺¹ X x)
```

On Expressivity

```
             fixed     floating
data X (a : A @i) (b : B) : ⋆ @j where
  C : … X a b @k
```

On Expressivity

```
             fixed    floating
data X (a : A @i) (b : B) : ⋆ @j where
  C : … X a b @k
```

- cumulativity + displacement help avoid code duplication
  ↪️ with exceptions: see the paper

On Expressivity

```
             fixed      floating
data X (a : A @i) (b : B) : ⋆ @j where
  C : … X a b @k
```

- cumulativity + displacement help avoid code duplication
  ↪️ with exceptions: see the paper
- ill-typed definitions fail to check at the same points
  ↪️ e.g. three type-theoretic paradoxes (in 3… 2… 1…)

is StraTT consistent?

we don't know.

- type-theoretic paradoxes failing
- StraTT w/o floating functions (subStraTT)

Three Type-Theoretic Paradoxes

1. **Russell's paradox**: an inductive containing inductives that don't contain themselves
2. **Burali-Forti's paradox**: a well-founded inductive strictly greater than itself
3. **Hurkens' paradox**: simplification of Girard's paradox

all fail to type check!

- trying to use higher-level term at lower level
- trying to use displaced term at undisplaced type

Consistency of subStraTT (StraTT w/o floating functions)

Theorem (logical consistency):
∄ closed, well-typed inhabitant of ⊥

Proof sketch:

1. define logical relation (interp types as sets of terms)
   ↳ interp. empty type as empty set
2. prove properties: cumulativity, conversion, bwd closure
3. show well typed terms inhabit interp. of their type

Consistency of StraTT?

Conjecture (logical consistency):
∄ closed, well-typed inhabitant of $\bot$

Proof sketch:

1. define logical relation (interp types as sets of terms)
   ↳ interp. empty type as empty set
2. prove properties: ~~cumulativity~~, conversion, bwd closure
   ↳ floating functions break preserving cumulativity
3. show well typed terms inhabit interp. of their type

Contributions

- StraTT: type theory w/ stratified judgements
- type safety (not presented)
- prototype implementation
  ↳ displacement
  ↳ datatypes
  ↳ annotation inference
- subStraTT consistency proof

Future Work

- **StraTT consistency**
- expressivity
  ↳ vs. TT + prenex lvl polymorphism
- inference sound/complete
- extensions
  ↳ actual level polymorphism

thank you!

for more:

- paper: https://doi.org/10.1007/978-3-031-91118-7_10
- proofs/implementation: github.com/plclub/StraTT

$\boxed{\Delta; \Gamma \vdash a :^k A}$ *(Typing)*

DT-Type
$$\frac{\Delta \vdash \Gamma}{\Delta; \Gamma \vdash \star :^k \star}$$

DT-Pi
$$\frac{\Delta; \Gamma \vdash A :^j \star \quad \Delta; \Gamma, x :^j A \vdash B :^k \star \quad j < k}{\Delta; \Gamma \vdash \Pi x :^j A.\, B :^k \star}$$

DT-Arrow
$$\frac{\Delta; \Gamma \vdash A :^k \star \qquad \Delta; \Gamma \vdash B :^k \star}{\Delta; \Gamma \vdash A \to B :^k \star}$$

DT-AbsTy
$$\frac{\Delta; \Gamma \vdash A :^j \star \quad \Delta; \Gamma, x :^j A \vdash b :^k B \quad j < k}{\Delta; \Gamma \vdash \lambda x.\, b :^k \Pi x :^j A.\, B}$$

DT-AppTy
$$\frac{\Delta; \Gamma \vdash b :^k \Pi x :^j A.\, B \quad \Delta; \Gamma \vdash a :^j A \quad j < k}{\Delta; \Gamma \vdash b\ a :^k B\{a/x\}}$$

DT-AbsTm
$$\frac{\Delta; \Gamma \vdash A :^k \star \quad \Delta; \Gamma \vdash B :^k \star \quad \Delta; \Gamma, x :^k A \vdash b :^k B}{\Delta; \Gamma \vdash \lambda x.\, b :^k A \to B}$$

DT-AppTm
$$\frac{\Delta; \Gamma \vdash b :^k A \to B \quad \Delta; \Gamma \vdash a :^k A}{\Delta; \Gamma \vdash b\ a :^k B}$$

$$\frac{\vdash \Delta \qquad \Delta; \varnothing \vdash A :^k \star \quad \Delta; \varnothing \vdash a :^k A \quad x \notin \operatorname{dom} \Delta}{\vdash \Delta, x :^k A \coloneqq a}$$

$$\frac{\Delta \vdash \Gamma \qquad \Delta; \Gamma \vdash A :^k \star \quad x \notin \operatorname{dom} \Gamma \quad x \notin \operatorname{dom} \Delta}{\Delta \vdash \Gamma, x :^k A}$$

DT-Var
$$\frac{x :^j A \in \Gamma \quad \Delta \vdash \Gamma \qquad j \leq k}{\Delta; \Gamma \vdash x :^k A}$$

DT-Const
$$\frac{x :^j A \coloneqq a \in \Delta \quad \Delta \vdash \Gamma \qquad \vdash \Delta \quad i + j \leq k}{\Delta; \Gamma \vdash x^i :^k A^{+i}}$$

DT-Bottom
$$\frac{\Delta \vdash \Gamma}{\Delta; \Gamma \vdash \bot :^k \star}$$

DT-Absurd
$$\frac{\Delta; \Gamma \vdash A :^k \star \quad \Delta; \Gamma \vdash b :^k \bot}{\Delta; \Gamma \vdash \mathrm{absurd}(b) :^k A}$$

DT-Conv
$$\frac{\Delta; \Gamma \vdash a :^k A \quad \Delta; \Gamma \vdash B :^k \star \quad \Delta \vdash A \equiv B}{\Delta; \Gamma \vdash a :^k B}$$

$\boxed{[\![A]\!]_k}$ $\boxed{a \in [\![A]\!]_k}$

$$\frac{}{[\![\star]\!]_k} \qquad \frac{}{[\![\bot]\!]_k} \qquad \frac{j < k \qquad [\![A]\!]_j \qquad \forall y.\, y \in [\![A]\!]_j \longrightarrow [\![B\{y/x\}]\!]_k}{[\![\Pi x :^j A.\, B]\!]_k} \qquad \frac{A \Rightarrow B \qquad [\![B]\!]_k}{[\![A]\!]_k}$$

$$A \in [\![\star]\!]_k \triangleq [\![A]\!]_k \qquad f \in [\![\Pi x :^j A.\, B]\!]_k \triangleq \forall y.\, y \in [\![A]\!]_j \longrightarrow f\ y \in [\![B\{y/x\}]\!]_k$$

$$a \in [\![\bot]\!]_k \triangleq \mathbf{0} \qquad a \in [\![A]\!]_k \triangleq a \in [\![B]\!]_k \quad (\mathit{where}\ A \Rightarrow B)$$