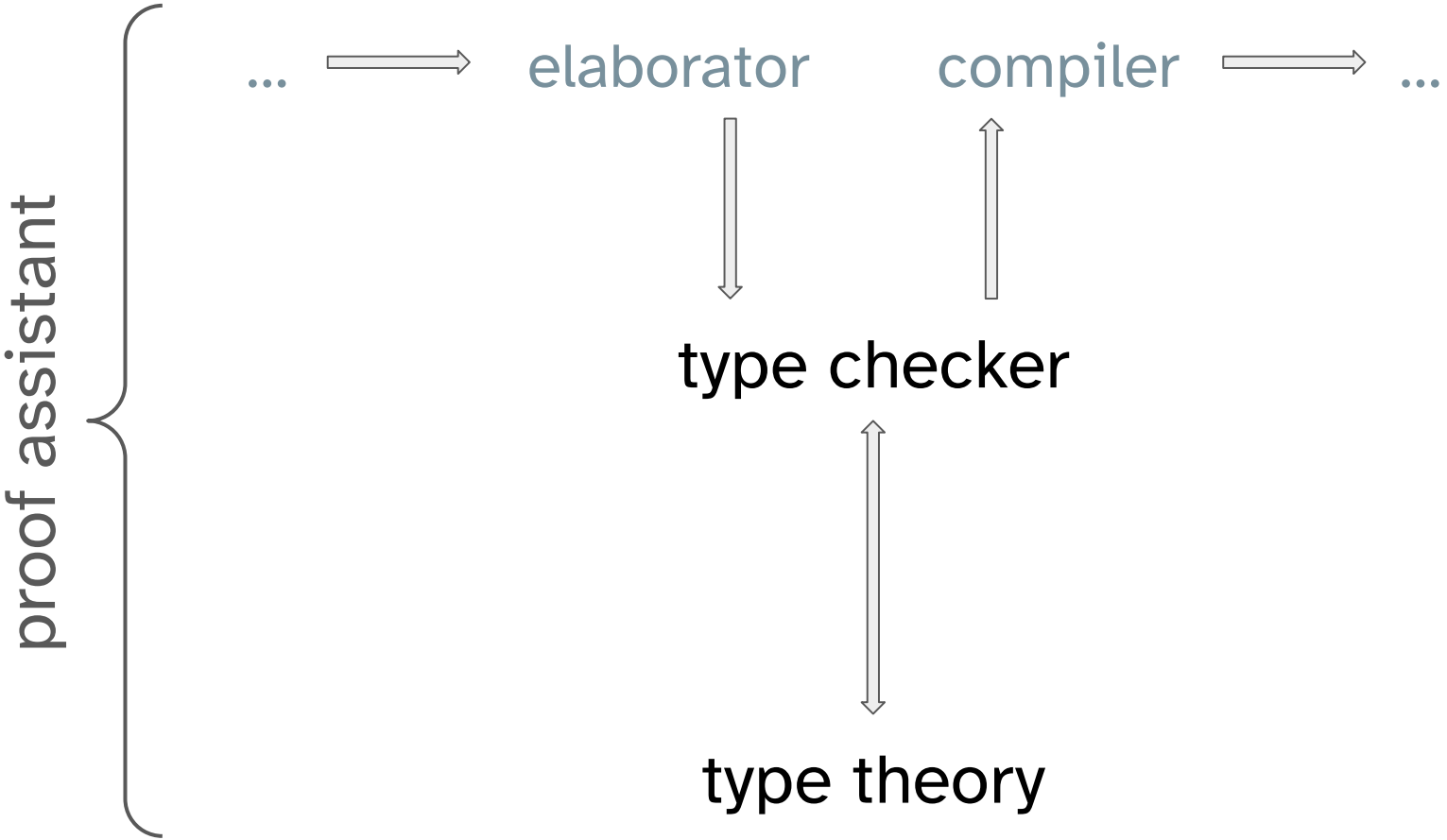
internalizing extensions in lattices of type theories

Jonathan Chan

21 November 2024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elaborator
compiler
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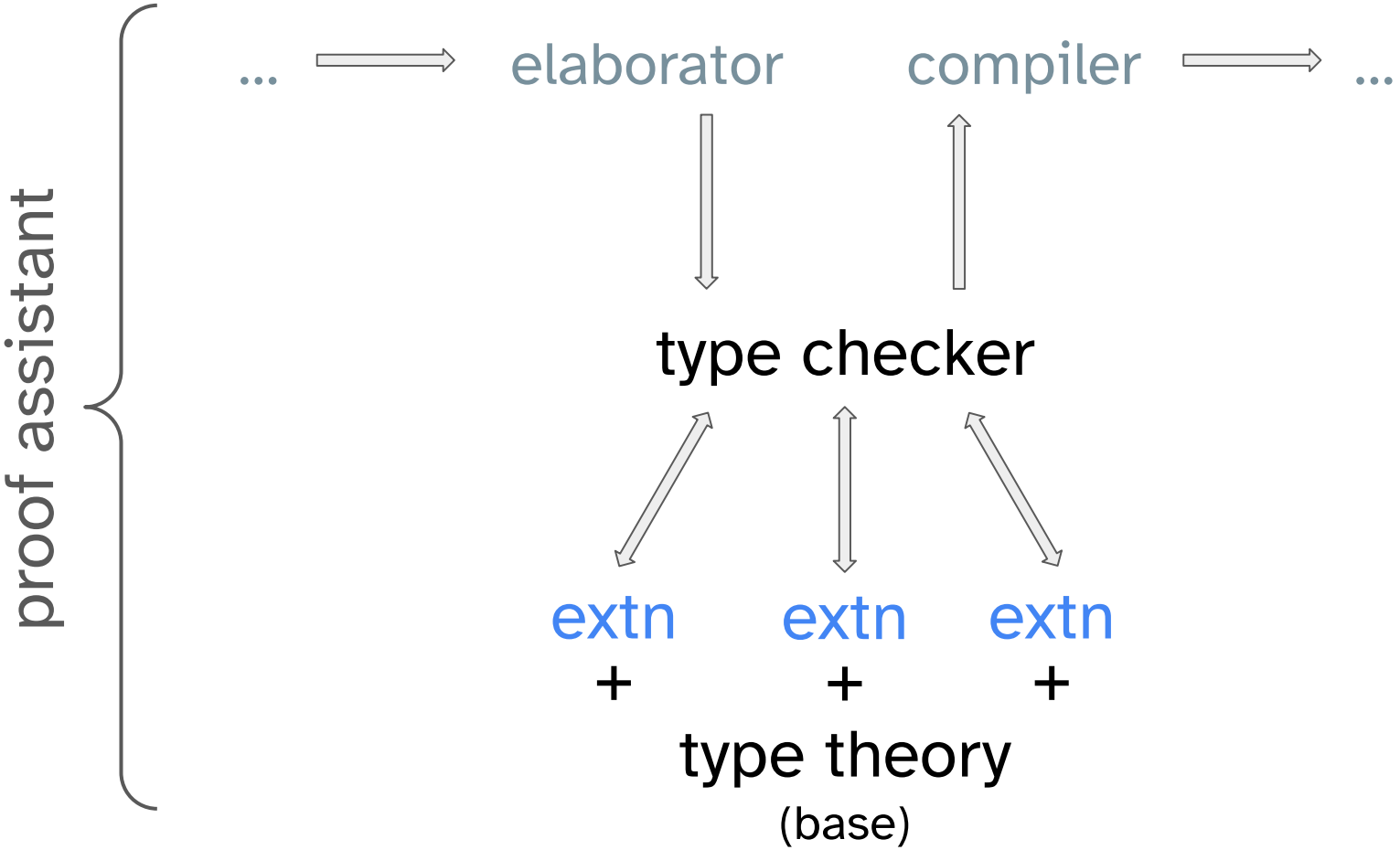
type checker
type theory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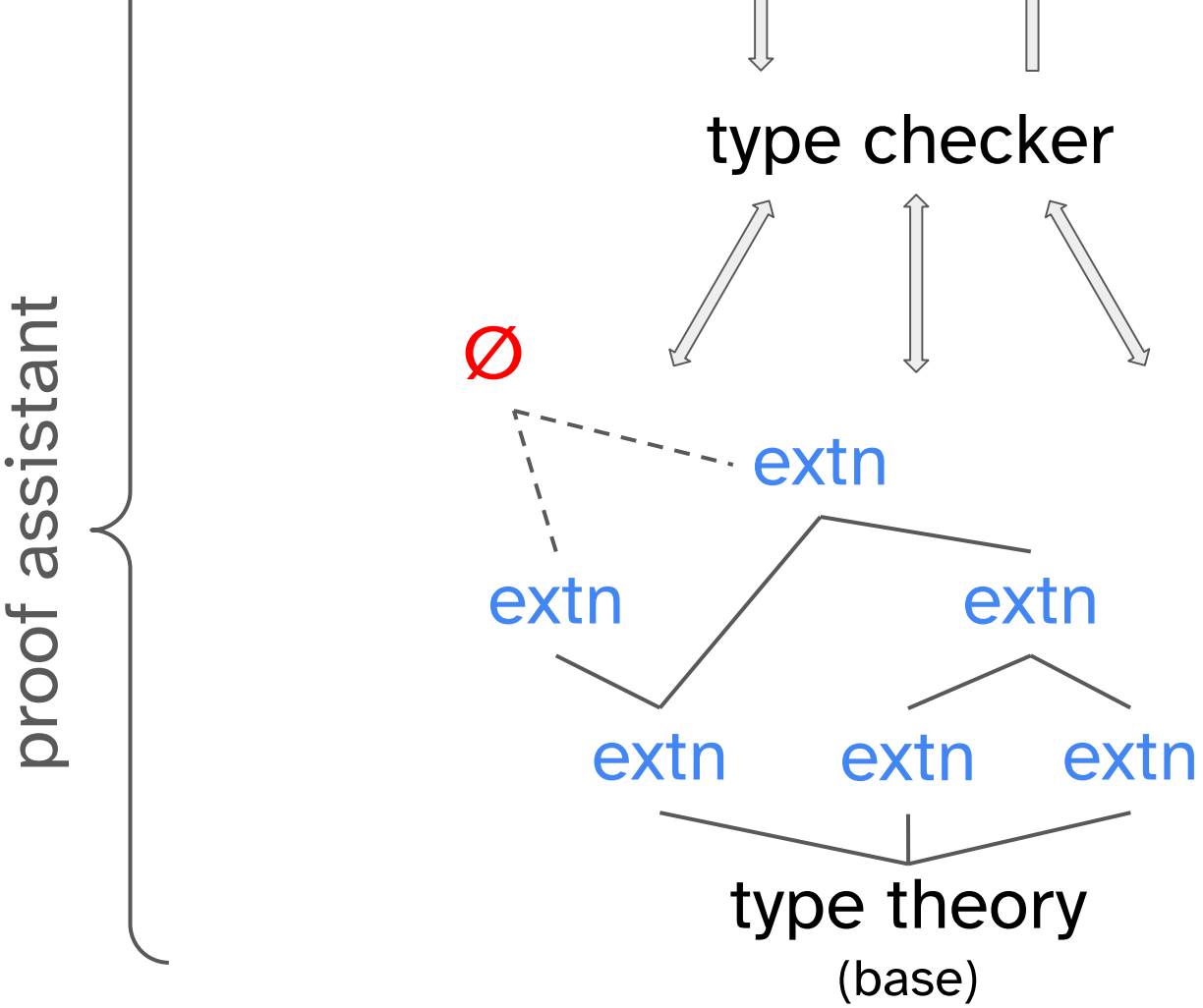
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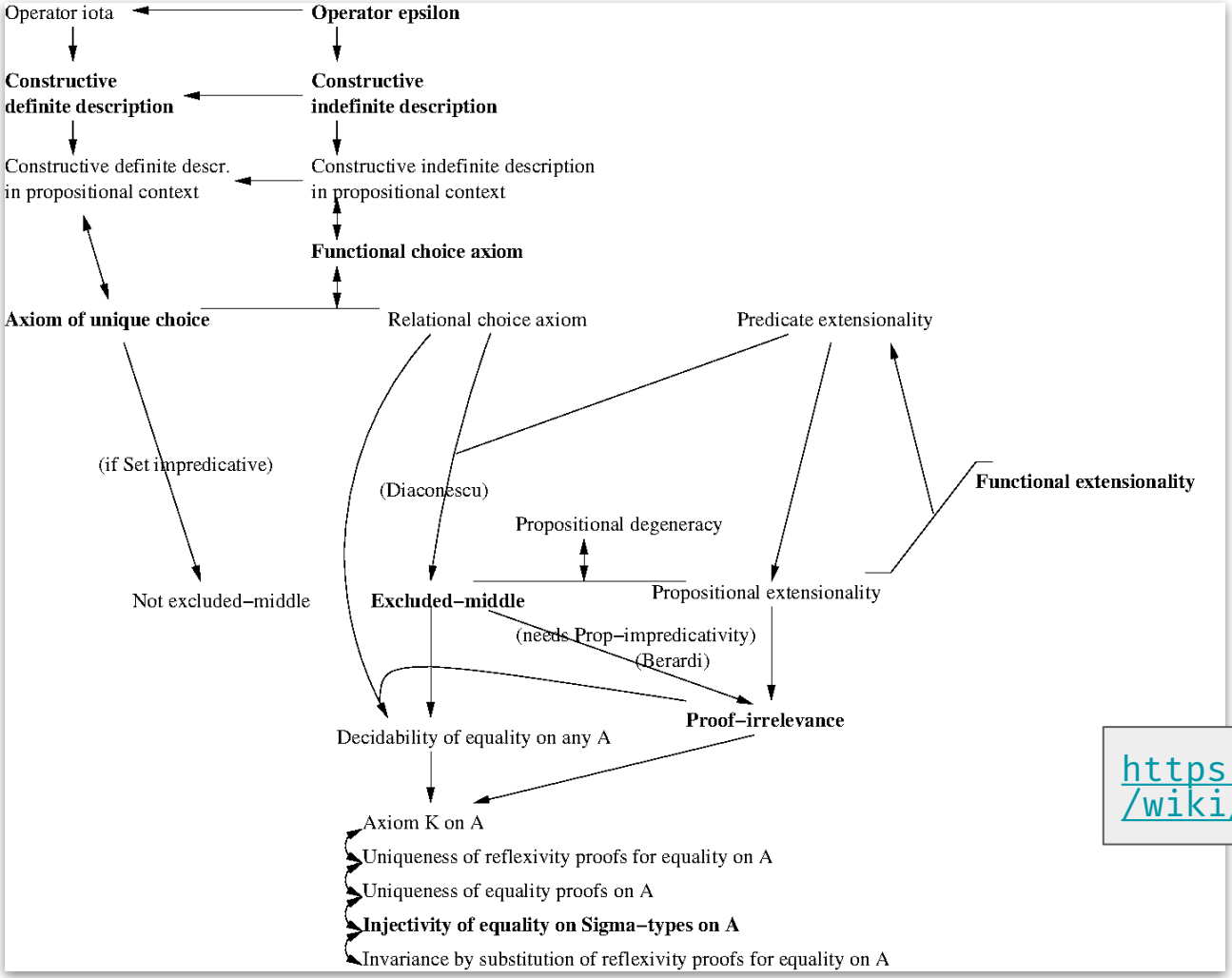


https://github.com/coq/coq/wiki/The-Logic-of-Coq

Checking options for consistency

Agda checks that options used in imported modules are consistent with each other.

An *infective* option is an option that if used in one module, must be used in all modules that depend on this module. The following options are infective:

- `--cohesion`
- `--erased-matches`
- `--erasure`
- `--flat-split`
- `--guarded`
- `--prop`
- `--rewriting`
- `--two-level`

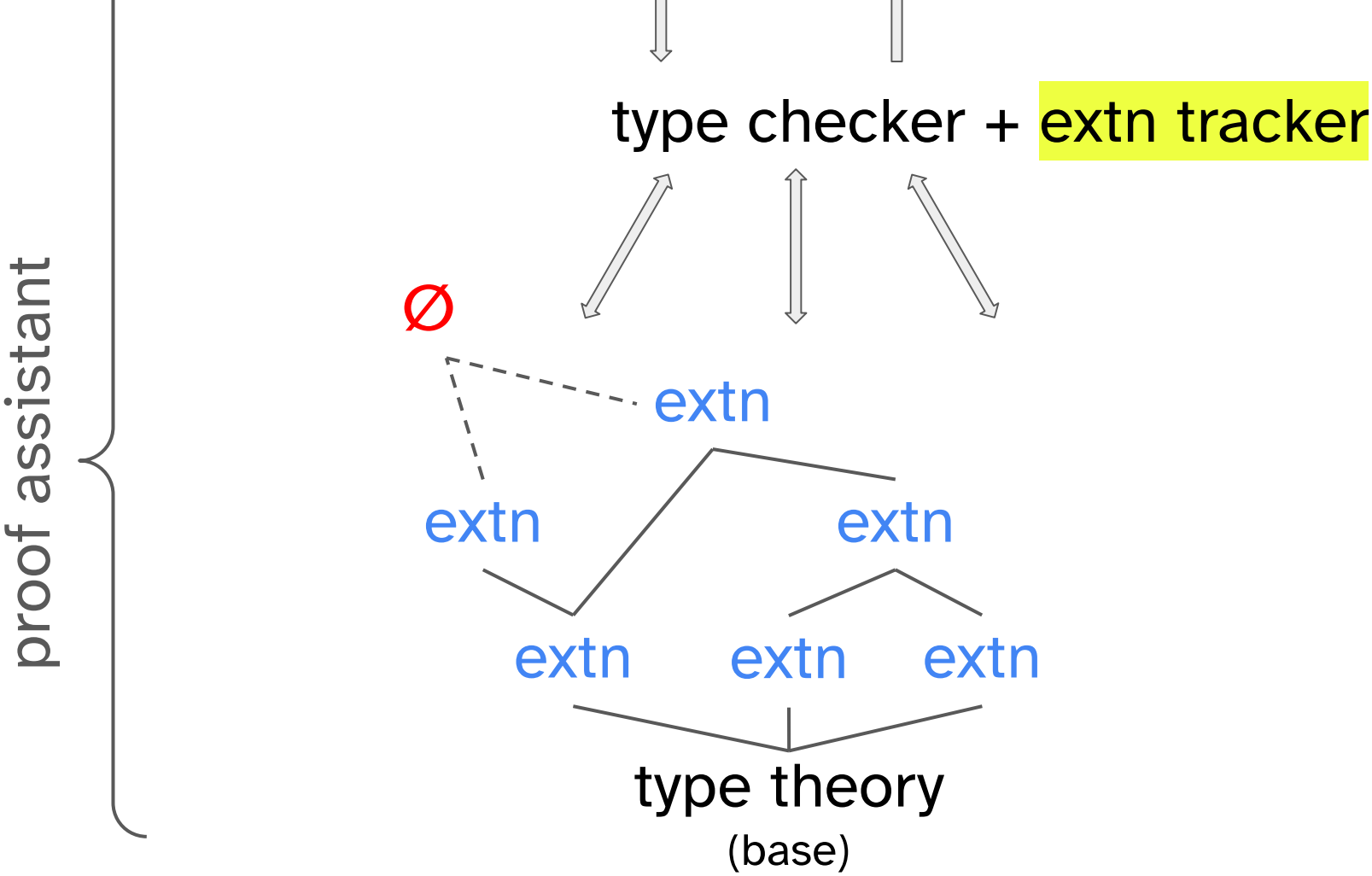
Furthermore `--cubical` and `--erased-cubical` are *jointly infective*: if one of them is used in one module, then one or the other must be used in all modules that depend on this module.

A *coinfective* option is an option that if used in one module, must be used in all modules that this module depends on. The following options are coinfective:

- `--safe`
- `--without-K`
- `--no-universe-polymorphism`
- `--no-sized-types`
- `--no-guardedness`
- `--level-universe`

Furthermore the option `--cubical-compatible` is mostly coinfective. If a module uses `--cubical-`

https://agda.readthedocs.io/en/latest/tools/command-line-options.html#checking-options-for-consistency


proof assistant
type checker + extn tracker
Ø
extn
extn
extn
extn
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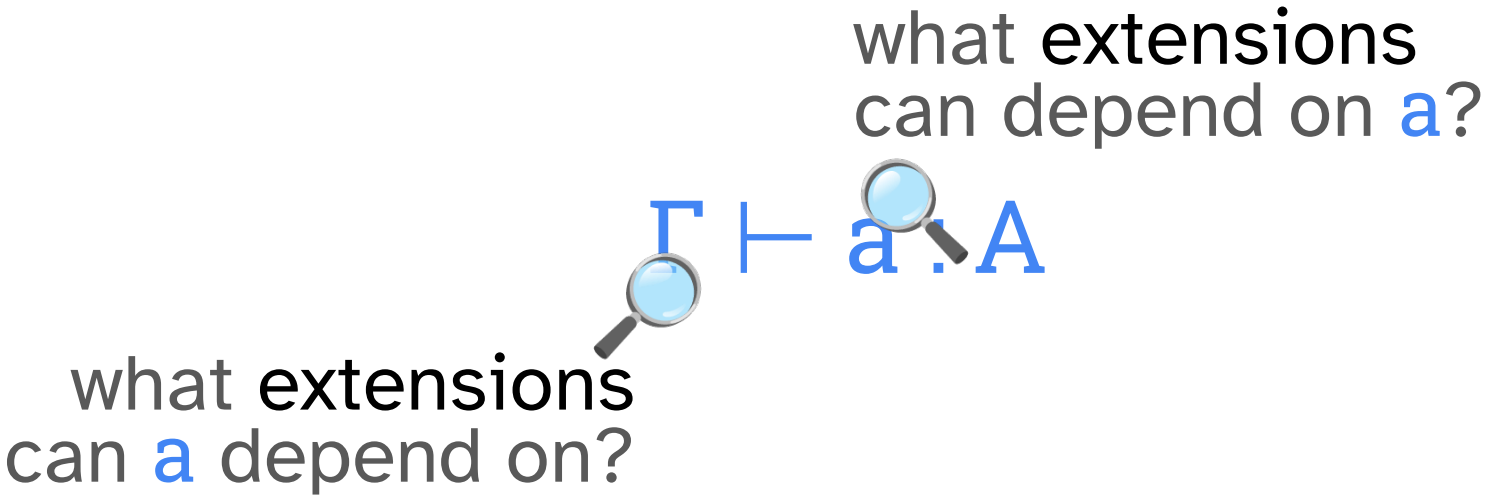
extn
type theory
(base)

what's missing: internalization

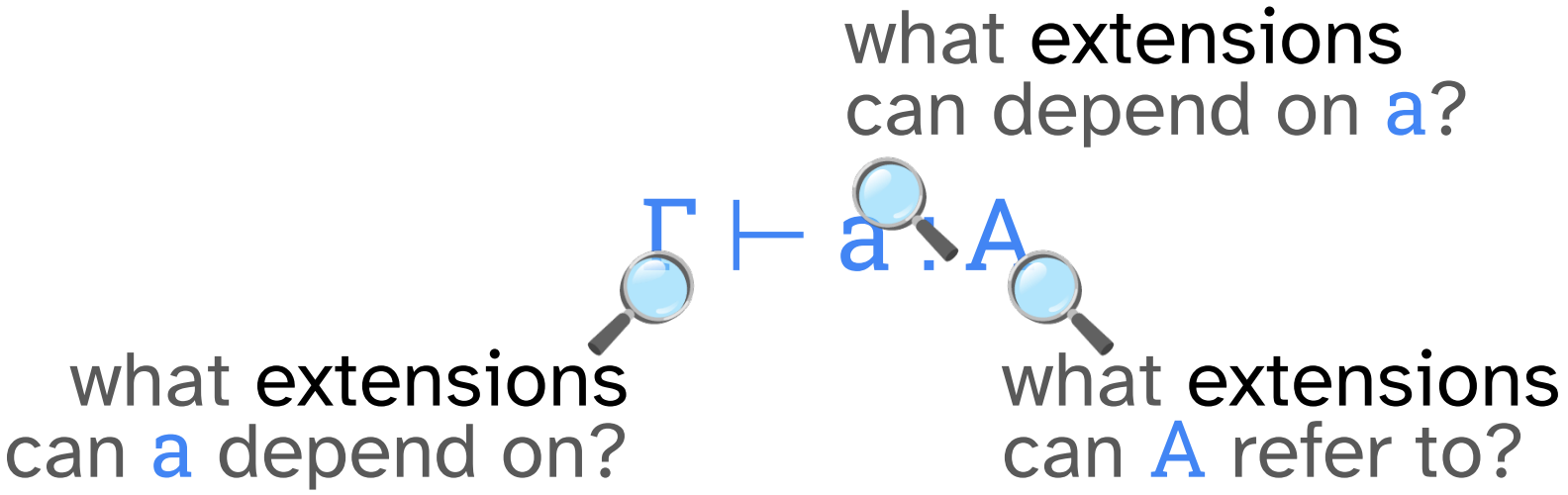
$\Gamma \vdash a : A$

what extensions
can a depend on?

what's missing: internalization



what's missing: internalization



Dependent Calculus of Indistinguishability (Liu, Chan, Shi, Weirich 2024/25)

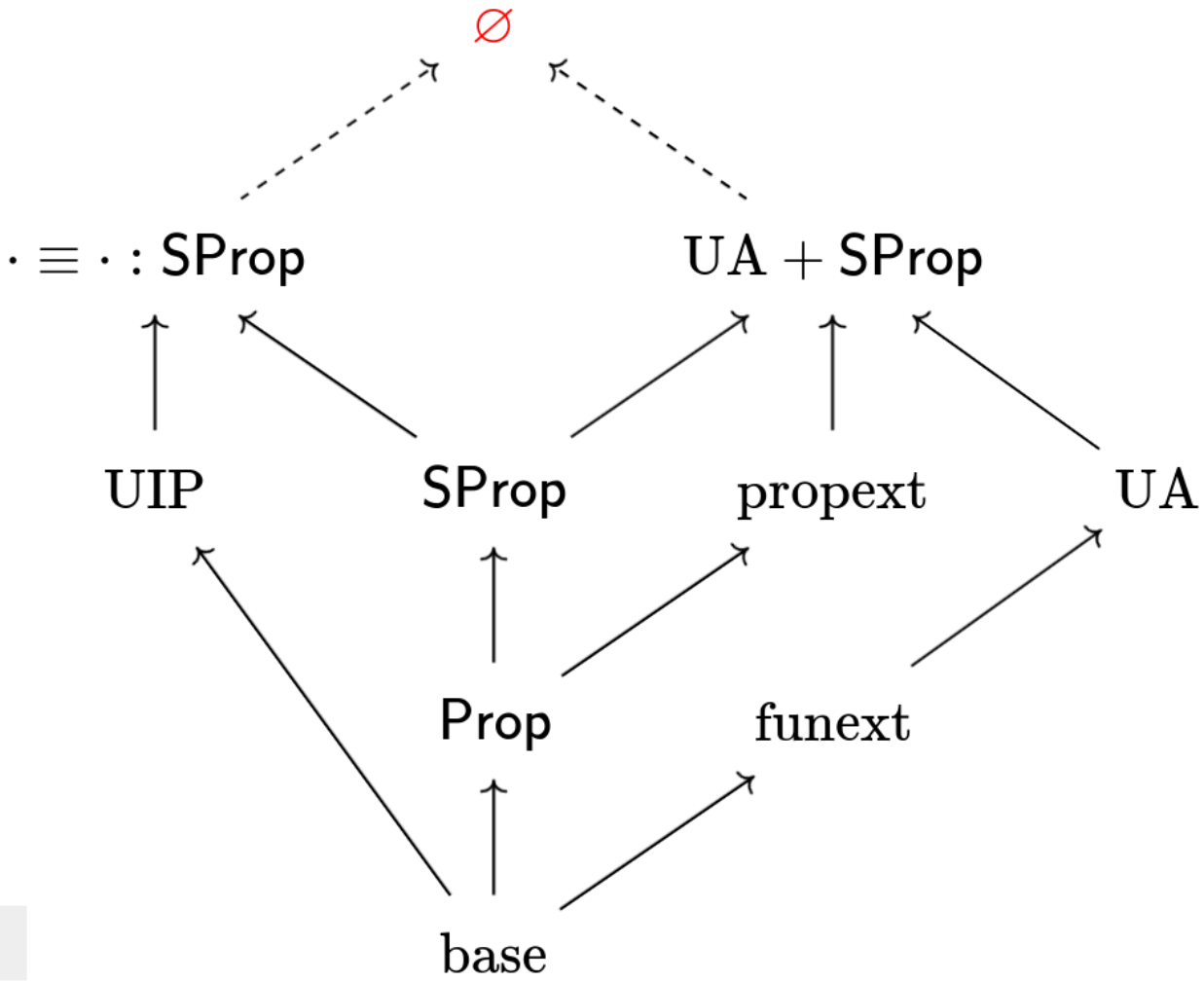
$$\Gamma, x : B \mathbin{@} \ell_1 \vdash a : A \mathbin{@} \ell_2$$

$\ell ::=$ levels from a lattice

outline

of the rest of the talk

1. extensions
2. DCOI
3. lattices of TTs
4. related work



1. extensions

univalence

$$(A \simeq B) \simeq (A \equiv B)$$

$$\mathrm{ua} : \forall (A\ B : \mathrm{Type}).\ (A \simeq B) \to (A \equiv B)$$

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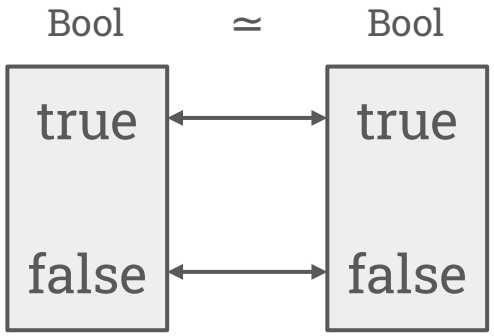

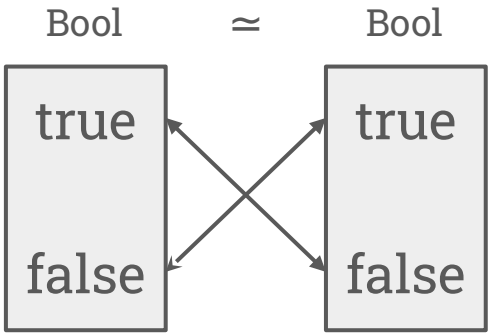

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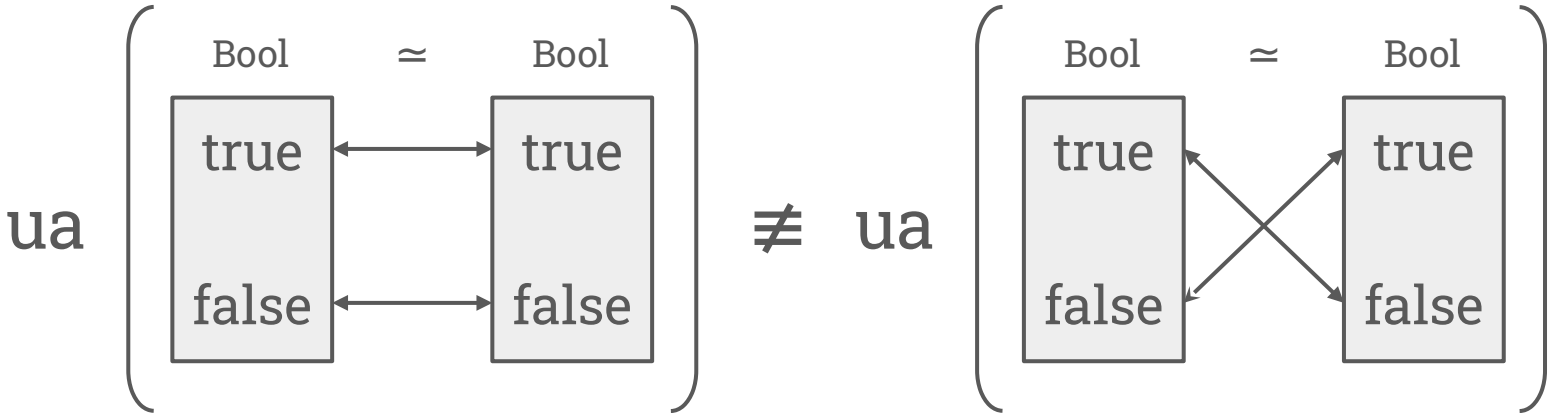


1. extensions

uniqueness of identity proofs

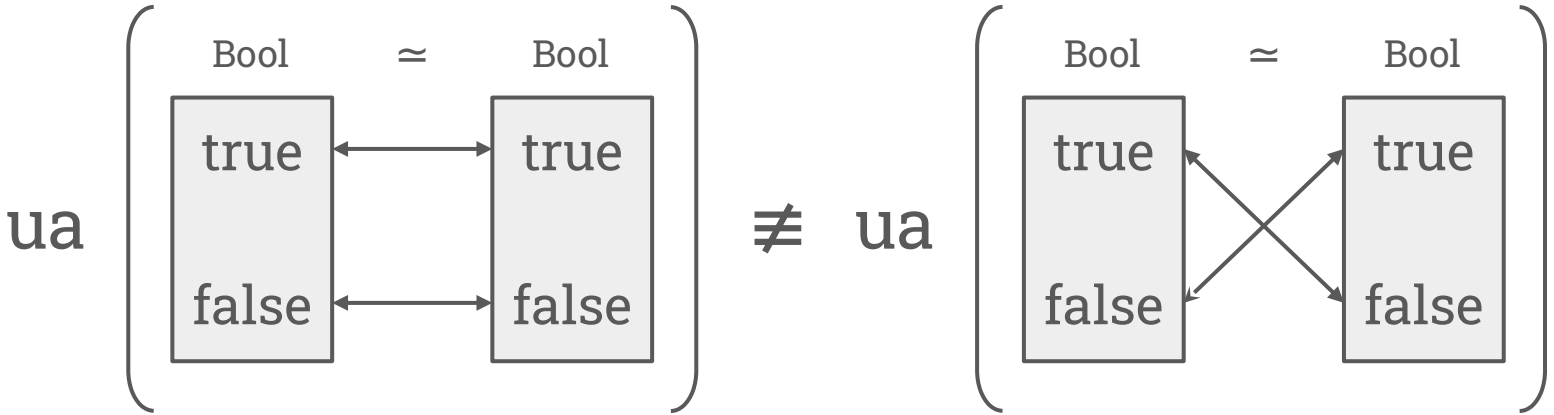
```
UIP : ∀(A : Type) (a b : A) (p q : a ≡ b). p ≡ q
UIP _ _ _ refl refl ⇝ refl

K : ∀ (A : Type) (a : A) (P : a ≡ a → Prop) (p : a ≡ a).
    P refl → P p
K _ _ _ refl d ⇝ d
```

1. extensions

UIP + univalence ⇒ ⊥

UIP Type Bool Bool : ∀(p q : Bool ≡ Bool). p ≡ q



1. extensions

DCOI

Dependent Calculus of Indistinguishability (Liu, Chan, Shi, Weirich 2024/25)

$$\Gamma, x : B \mathbin{@} \ell_1 \vdash a : A \mathbin{@} \ell_2$$

$$\ell ::= L \leq H \leq S$$

DCOI for information flow control

$$\frac{x : A \mathbin{@} \ell \in \Gamma \qquad \ell \leq \ell'}{\Gamma \vdash x : A \mathbin{@} \ell'}$$

$$\lambda x^{L}\, y^{H}.\; x : (\mathrm{Nat} \mathbin{@} L) \to (\mathrm{Nat} \mathbin{@} H) \to \mathrm{Nat} \mathbin{@} L$$

DCOI for information flow control

$$\frac{\Gamma \vdash a : A \mathbin{@} \ell \qquad \ell \leq \ell'}{\Gamma \vdash a : A \mathbin{@} \ell'}$$

$$\lambda x^{L}\, y^{H}.\, x : (\mathrm{Nat} \mathbin{@} L) \to (\mathrm{Nat} \mathbin{@} H) \to \mathrm{Nat} \mathbin{@} L$$

$$\lambda x^{L}\, y^{H}.\, y : (\mathrm{Nat} \mathbin{@} L) \to (\mathrm{Nat} \mathbin{@} H) \to \mathrm{Nat} \mathbin{@} H$$

$$\lambda x^{L}\, y^{H}.\, x + 1 : (\mathrm{Nat} \mathbin{@} L) \to (\mathrm{Nat} \mathbin{@} H) \to \mathrm{Nat} \mathbin{@} H$$

DCOI for run-time erasure of types

$$\lambda A^{H}\ x^{L}\ y^{H}.\ x : \Pi(A : \mathrm{Type}_1 \ @\ H).\ (A\ @\ L) \to (A\ @\ H) \to A\ @\ L$$

DCOI for run-time erasure of types

$$\frac{\Gamma \vdash a : A \mathbin{@} \ell}{\exists \ell', \mathcal{U}.\, \Gamma \vdash A : \mathcal{U} \mathbin{@} \ell'}$$

$$\lambda A^{H}\, x^{L}\, y^{H}.\, x : \Pi(A : \mathrm{Type}_1 \mathbin{@} H).\, (A \mathbin{@} L) \to (A \mathbin{@} H) \to A \mathbin{@} L$$

$$\Pi(A : \mathrm{Type}_1 \mathbin{@} H).\, (A \mathbin{@} L) \to (A \mathbin{@} H) \to A : \mathrm{Type}_2 \mathbin{@} H$$

DCOI for compile-time irrelevance

$$\lambda P^{H}\ p^{L}.\ p : \Pi(P : (\mathrm{Nat} \mathbin{@} S) \to \mathrm{Type} \mathbin{@} H).\ (P\ 0^{S} \mathbin{@} L) \to P\ 1^{S} \mathbin{@} L$$

DCOI for compile-time irrelevance

$$\frac{\Gamma \vdash f = g \mathbin{@} \ell \qquad \ell' \not\leq \ell}{\Gamma \vdash f\ a^{\ell'} = g\ b^{\ell'} \mathbin{@} \ell}$$

$$\vdash P\ 0^{S} = P\ 1^{S} \mathbin{@} H$$

$$\lambda P^{H}\ p^{L}.\ p : \Pi(P : (\mathrm{Nat} \mathbin{@} S) \to \mathrm{Type} \mathbin{@} H).\ (P\ 0^{S} \mathbin{@} L) \to P\ 1^{S} \mathbin{@} L$$

DCOI internalizes indistinguishability

$$\frac{\Gamma \vdash a = b \mathbin{@} \ell}{\Gamma \vdash \mathrm{refl} : a \equiv^{\ell} b \mathbin{@} \ell'}$$

$$\vdash P\ 0^{S} = P\ 1^{S} \mathbin{@} H$$

$$\lambda P^{H}.\ \mathrm{refl} : \Pi(P : (\mathrm{Nat} \mathbin{@} S) \to \mathrm{Type} \mathbin{@} H).\ P\ 0^{S} \equiv^{H} P\ 1^{S} \mathbin{@} L$$

DCOI and eliminating ⊥

$$\mathrm{head} : \forall(l : \mathrm{List}\ A\ @\ \mathrm{L}).\ (0 < \mathrm{len}\ l\ @\ \mathrm{H}) \to A\ @\ \mathrm{L}$$
$$\mathrm{head}\ \mathrm{nil}^{\mathrm{L}} : (0 < 0\ @\ \mathrm{H}) \to A\ @\ \mathrm{L}$$

DCOI and eliminating ⊥

$$\frac{\Gamma \vdash b : \bot \mathbin{@} \ell}{\Gamma \vdash \mathsf{absurd}\ b : A \mathbin{@} \ell'}$$

$$\mathsf{zeroNotGt} : (0 < 0 \mathbin{@} H) \to \bot \mathbin{@} H$$

$$\mathsf{head} : \forall (l : \mathsf{List}\ A \mathbin{@} L).\ (0 < \mathsf{len}\ l \mathbin{@} H) \to A \mathbin{@} L$$

$$\mathsf{head}\ \mathsf{nil}^{L} : (0 < 0 \mathbin{@} H) \to A \mathbin{@} L$$

$$\mathsf{head}\ \mathsf{nil}^{L} \coloneqq \lambda q^{H}.\ \mathsf{absurd}\ (\mathsf{zeroNotGt}\ q)$$

variables

$$\frac{x : A \mathbin{@} \ell \in \Gamma \qquad \ell \leq \ell'}{\Gamma \vdash x : A \mathbin{@} \ell'}$$

subsumption

$$\frac{\Gamma \vdash a : A \mathbin{@} \ell \qquad \ell \leq \ell'}{\Gamma \vdash a : A \mathbin{@} \ell'}$$

regularity

$$\frac{\Gamma \vdash a : A \mathbin{@} \ell}{\exists \ell', \mathcal{U}.\, \Gamma \vdash A : \mathcal{U} \mathbin{@} \ell'}$$

$$\frac{\Gamma \vdash f = g \mathbin{@} \ell \qquad \ell' \not\leq \ell}{\Gamma \vdash f\ a^{\ell'} = g\ b^{\ell'} \mathbin{@} \ell}$$

indistinguishability

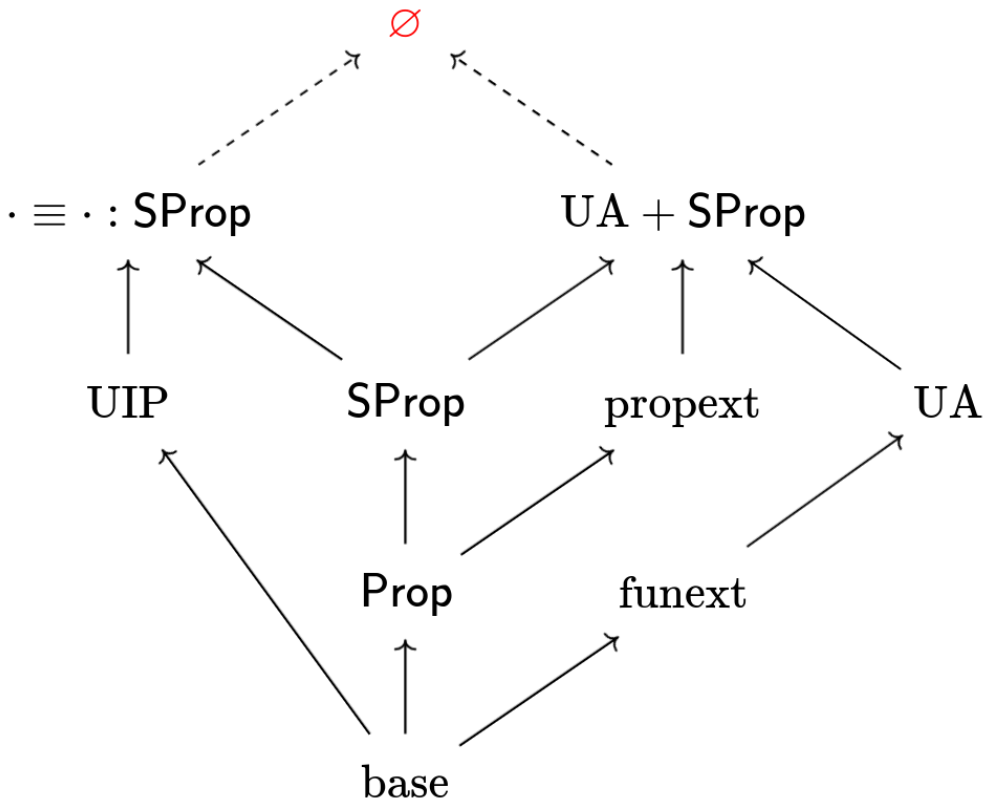
$$\frac{\Gamma \vdash a = b \mathbin{@} \ell}{\Gamma \vdash \mathrm{refl} : a \equiv^{\ell} b \mathbin{@} \ell'}$$

equality

$$\frac{\Gamma \vdash b : \bot \mathbin{@} \ell}{\Gamma \vdash \mathrm{absurd}\ b : A \mathbin{@} \ell'}$$

$\bot$ elimination

lattices of type theories



$\ell ::= \mathsf{base} \mid \mathsf{Prop} \mid \mathsf{SProp} \mid \mathsf{funext} \mid \mathsf{propext} \mid \mathsf{UIP} \mid \mathsf{UA} \mid \ldots$

extensions at levels

$\Gamma \vdash \mathrm{Prop} : \mathrm{Type}_1 \ @\ \mathrm{Prop}$

$\Gamma \vdash \Pi(x : A).\ B : \mathrm{Prop} \ @\ \mathrm{Prop}$

$\Gamma \vdash K\ P\ p\ d : P\ p \ @\ \mathrm{UIP}$

$K\ P\ \mathrm{refl}\ d \rightsquigarrow d$

$\Gamma \vdash \mathrm{propext} : \forall(A\ B : \mathrm{SProp}).\ (A \leftrightarrow B) \to A \equiv B \ @\ \mathrm{propext}$

example: indistinguishability

$$\mathrm{UIP} \not\leq \mathrm{UA}$$

$$\vdash \mathrm{f}\ \mathrm{x}^{\mathrm{UIP}} = \mathrm{f}\ \mathrm{y}^{\mathrm{UIP}} @ \mathrm{UA}$$

example: indistinguishability + equality

$$\mathrm{UIP} \not\leq \mathrm{UA}$$

$$\vdash \mathrm{f}\ \mathrm{x}^{\mathrm{UIP}} = \mathrm{f}\ \mathrm{y}^{\mathrm{UIP}} @ \mathrm{UA}$$

$$\vdash \mathrm{refl} : \mathrm{f}\ \mathrm{x}^{\mathrm{UIP}} \equiv^{\mathrm{UA}} \mathrm{f}\ \mathrm{y}^{\mathrm{UIP}} @ \ell$$

example: regularity

$$(\ldots \text{ua} \ldots, \lambda q. \ldots) :$$
$$\Sigma(p : \text{Bool} \equiv^{\text{UA}} \text{Bool} \mathbin{@} \text{UA}).$$
$$[(p \equiv^{\text{UA}} \text{refl} \mathbin{@} \text{UA}) \to \bot \mathbin{@} \text{UA}] \mathbin{@} \text{base}$$

example: regularity + subsumption

$$(\ldots \text{ua} \ldots, \lambda q. \ldots) :$$
$$\Sigma(p : \text{Bool} \equiv^{\text{UA}} \text{Bool} \ @ \ \text{UA}).$$
$$[(p \equiv^{\text{UA}} \text{refl} \ @ \ \text{UA}) \to \bot \ @ \ \text{UA}] \ @ \ \text{UIP}$$

N.B. $\bot$ elimination: $\exists \ell$. inconsistent $\Rightarrow \forall \ell$. inconsistent

$$\frac{\vdash b : \bot \ @ \ \mathbf{UA + UIP}}{\vdash \mathrm{absurd}\ b : \bot \ @ \ \ell}$$

questions to answer

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1. expressivity: what extensions fit?
    - new rules/eliminators/universes/axioms/equalities

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2. expressivity: what more can be proven?
    - UIP facts about UA? constructive facts about classical?

questions to answer

1. expressivity: what extensions fit?
    - new rules/eliminators/universes/axioms/equalities
2. expressivity: what more can be proven?
    - UIP facts about UA? constructive facts about classical?
3. how to prove logical consistency?
    - extensibility to more extensions

part 1: implementation

- many common extensions
- reimplement parts of stdlibs of Rocq/Agda/Lean
- Agda 2LTT lib: UA + UIP
- level inference? level polymorphism?

part 2: formalization

- one (1) extension + base
- DCOI^{ω}: DCOI w/ predicative universes
- syntactic logical relation indexed by universe levels
- UIP/funext likely hold semantically
- impredicativity/typed equality likely difficult

part 2: formalization

- one (1) extension + base

- DCOI$^{\omega}$: DCOI w/ predicative universes
- syntactic logical relation indexed by universe levels

- syntactic modelling:
  type preserving translation to a consistent theory
- SProp [Gilbert et al., 2019], Ghost TT [Winterhalter, 2024]

related work

prior work

DCOI$^{\omega}$ *Consistency of a Dependent Calculus of Indistinguishability*. Yiyun Liu, Jonathan Chan, and Stephanie Weirich. POPL 2025.

DCOI *Internalizing Indistinguishability with Dependent Types*. Yiyun Liu, Jonathan Chan, Jessica Shi, and Stephanie Weirich. POPL 2024.

StraTT *Stratified Type Theory*. Jonathan Chan and Stephanie Weirich. ESOP 2025.

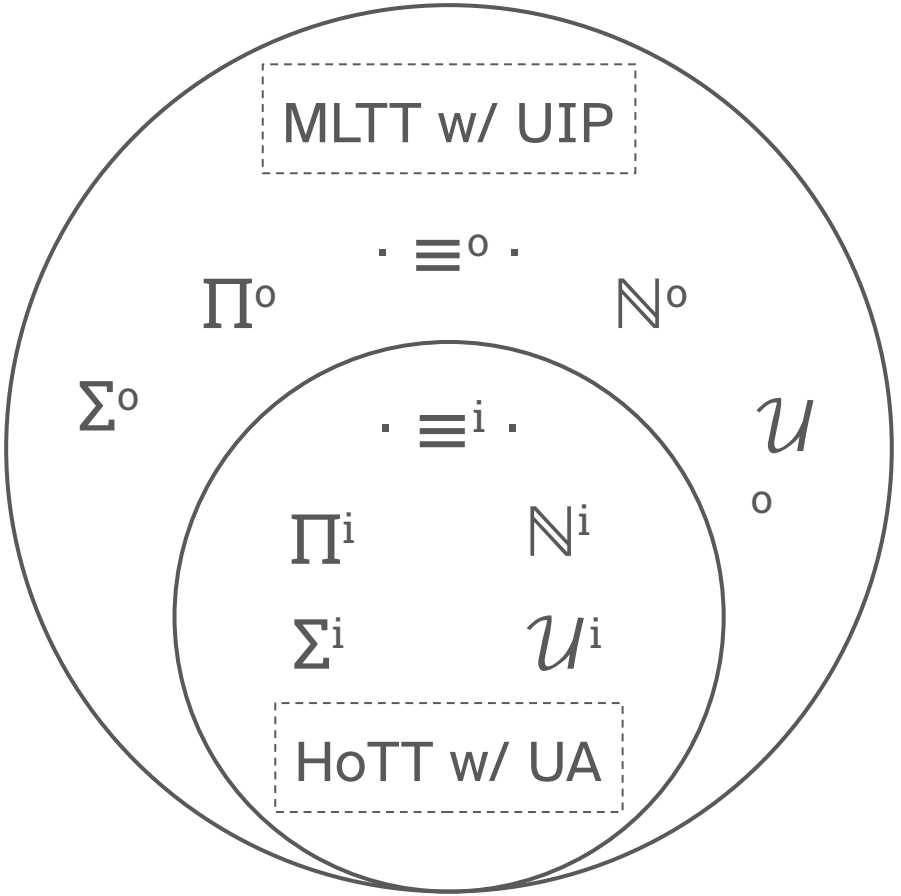
2LTT [Voevodsky; Altenkirch et al., 2016; Annenkov et al., 2023]

2LTT:

$$\forall x^{i}\ y^{i}.\ \uparrow x \equiv^{o} \uparrow y \to x \equiv^{i} y$$

DCOI:

$$\forall x^{L}\ y^{L}.\ x \equiv^{H} y \to x \equiv^{L} y$$



Trellys: *logical proofs + (diverging) programming*

λ^{θ}, LF^{θ}, PCC^{θ} [Casinghino, 2014]
Zombie [Sjöberg, 2015]

Sep^{3} [Kimmell et al., 2012]

Nax [Ahn, 2014]

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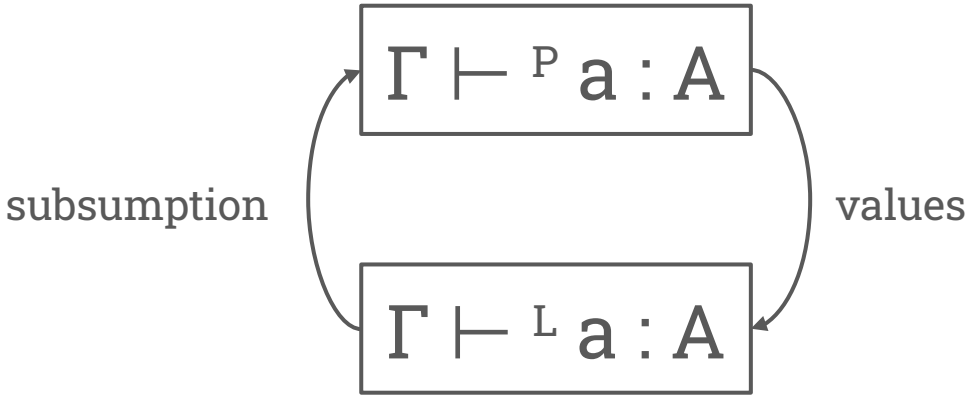
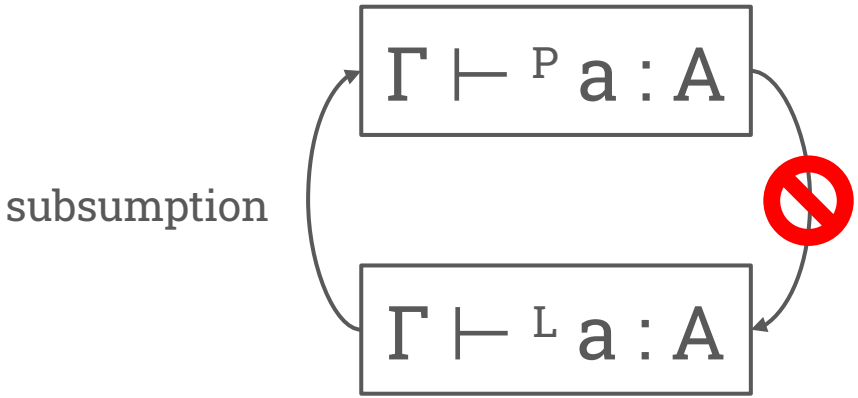


System DE [Liu and Weirich, 2023]



more proof assistants!

- **F⋆** has monadic ordered effects, including Tot ≤ Dv; Tot types cannot depend on Dv values
- **Idris 2** has pragmas total ≤ covering ≤ partial; partial terms will not reduce in types

Summary

- proof assistants have extensions
  that extend the underlying type theory
- some extensions are mutually incompatible
- type checkers track which extensions are used externally
- can’t refer to extensions internally —
  imprecise and restrictive
- DCOI: dependency tracking system w/ dependent types
- we might be able use DCOI to track extensions